

Trading Volume and Information Revelation in Stock Markets

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Abstract

I consider a market microstructure model in which the rates of public and private information arrival are probabilistic. The latter depends on the availability of private information that is stochastically changing over time. In equilibrium, traders estimate the availability of private information using past periods' trading volume and use this information to adjust their strategies. The time-series properties include contemporaneous correlation between price variability and volume and autocorrelation in price variability (similar to GARCH). The model explains why trading volume contains useful information for predicting volatility and provides predictions on the limit and market order placement strategies of traders.

I. Introduction

The academic literature in finance contains few theoretical papers on the informational role of trading volume, despite the common use of such information by practitioners and several stylized facts relating trading volume and asset prices. For example, Karpoff (1987) documented that stock return volatility and contemporaneous trading volume are positively correlated, and Lamoureux and Lastrapes (1990) find that trading volume in stock markets contains relevant information for predicting future volatility. In this paper, I develop a theoretical model of stock markets that is consistent with these and several other stylized facts on the stock return trading volume relation, and in which trading volume plays an important role in traders' learning. The model generates several interesting results concerning the behavior of trading volume and stock price variability.

In asset markets, the very source of uncertainty in the asset returns is constantly changing. In the case of equity returns, it may be that at a given point

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in time, the uncertainty over returns stems from uncertainty over the price of the firm's output. At other times, uncertainty over the launching of a new product, the profitability of a recent acquisition, or even the outcome of a takeover battle may be the prime sources of fluctuations in equity returns. If some of these sources of uncertainty are easier for traders to monitor than others, then it is reasonable to assume that there are changes in the availability of private information.

In my model, new private information about equity returns is available in any given period only with some probability. In addition, this probability changes stochastically over time as the source of uncertainty in equity returns changes. The public information arrival is also probabilistic, but, for simplicity, its arrival rate is constant. There are two types of traders: informed speculators and liquidity traders. Both types of traders act competitively. The market is organized as a limit order market.

In equilibrium, the informed speculators compete in trading with liquidity traders, and trade aggressively as soon as they receive new private information. Their trading soon reveals their private information to other market participants and leads the other traders to revise their estimates for both the value of the asset and the availability of private information. The latter affects traders' behavior: As the probability of the existence of private information increases, liquidity traders become wary and start posting more conservative limit orders. Initially, a higher probability of private information attracts more speculators to search for information, so the number of informed traders increases. However, since this influx makes liquidity traders more cautious, the number of informed traders may later decrease.

In my model, traders estimate the availability of private information using past periods' trading volume and use this information to adjust their strategies. This accords with the observation that the information contained in trading volume is important for traders' learning and affects their behavior. In this respect, the paper is closely related to Blume et al. (1994) and Bernardo and Judd (1999). These papers develop models in which traders use previous periods' trading volume to make inferences about the quality of informed traders' signals, which is important for estimating the payoff to the security. In many other models on the role of trading volume, such as Campbell et al. (1993) and Wang (1994), trading volume can help an econometrician learn about the expected returns on the risky asset. Yet, in contrast to the findings in this paper and Blume et al. (1994) and Bernardo and Judd (1999), the traders themselves do not learn anything from trading volume.

My model generates several results related to the stock price variability trading volume relation. First, consistent with empirical research, e.g., in Karpoff (1987) and Gallant et al. (1992), there is positive correlation between price variability and volume and autocorrelation in price variability. Positive correlation between price variability and trading volume arises because trading by informed traders reveals private information to markets and affects prices. The expected price variability depends on the availability of private information, and inherits any autocorrelation in the process that determines it. If, as I assume, the availability of private information changes according to a two-state Markov process, the autocorrelation function for price variability is positive and geometrically de-

caying. This is similar to the autocorrelation functions of price variability for individual stocks and stock indices documented in the empirical research (see, e.g., Brock et al. (1996)). Due to changes in traders' behavior as they learn about the state of the economy, trading volume can be either positively or negatively autocorrelated.

My model predicts that the expected price variability, conditional on the public information set, is autocorrelated and mean reverting. In fact, I derive a closed-form solution to conditional variance of price changes that looks very similar to a GARCH (Generalized Autoregressive Conditionally Heteroskedastic) model. Interestingly, and in contrast to most GARCH models, the evolution of conditional variance in my model depends on trading volume. Another result is that the expected trading volume can be either positively or negatively correlated with the expected price variability. Finally, my model provides predictions on the limit and market order placement strategies of traders.

A few other theoretical papers address the above empirical regularities. Foster and Viswanathan (1995) build a speculative trading model, with changing expected volatility of the underlying asset returns, which can account for the positive correlation between volume and volatility. Harris and Raviv (1993) build a model based on differences in opinion that can account for the positive volume-absolute price change correlation and positive autocorrelation in volume. One drawback of the Harris and Raviv (1993) model is, however, that it concerns solely the arrival of public information, whereas, in reality, it is often difficult to relate the changes in volatility and volume to the arrival of any public information (see, e.g., Cutler et al. (1989)). Other theoretical articles that have addressed the issue include Brock and LeBaron (1996), de Fontnouvelle (1996), and Shalen (1993). Another, more empirically oriented approach to explaining these phenomena is the Mixture of Distribution Hypothesis pioneered by Clark (1973), and recently extended by Andersen (1996). In these models, the dynamic features are governed by information flow, modeled as a stochastic volatility process. In many ways, this paper is a theoretical extension of their work.¹

The paper is organized as follows. Section II describes the model. Section III characterizes an equilibrium where the informed traders trade aggressively whenever they receive private information. In Section IV, I study the time-series properties of this equilibrium. Section V concludes.

II. Basic Model

There are T periods, which can be thought of as days, a safe asset, and a single risky asset. The return on the safe asset is normalized to zero. The risky asset is the equity of a firm. The firm engages in two different activities over time, A and B , but only one activity at a time. The state of the economy is the firm's activity $s_t \in \{A, B\}$, which changes randomly over time as follows: A becomes B with probability μ and B becomes A with probability ε . The initial state is A with probability $\varepsilon/(\varepsilon + \mu)$. Both activities yield an identically distributed random

¹Other closely related articles include Bollerslev, Chou, and Kroner (1992), DeGennaro and Shrieves (1997), Dutta and Madhavan (1998), Grundy and McNichols (1989), Jones et al. (1994), Kandel and Pearson (1995), Kim and Verrecchia (2000), and Seppi (1992).

return $\delta_t \in \{-1, 1\}$ per period, where the probability that $\delta_t = 1$ is $\frac{1}{2}$. The only difference between the two activities is in the traders' ability to monitor them.

Both μ and ε are assumed to be less than one-half. Assuming that both ε and μ are equal to one-half would mean that the firm's activity is randomly determined each day. The assumption that ε and μ are both less than one-half thus requires some persistence in the firm's activity. This assumption is consistent with my interpretation of the changes in the source of uncertainty: takeover battles, price wars in output markets, or the launching of a new product are all examples of sources of uncertainty that tend to last for several days at a time.

The payoff to the risky asset in period T , F_T , is the sum of a fixed value $\bar{F} > 0$, and the periodic returns,

$$F_T = \bar{F} + \sum_{t=1}^T \delta_t.$$

Both public and private signals may reveal δ_t to the traders at time t . First, there is a public announcement at the end of period t with probability ρ , a_t , that reveals δ_t . With probability $(1 - \rho)$, there is no public announcement. For the trading volume to contain information on future volatility beyond price changes, it is necessary that $0 < \rho < 1$.

There are two types of agents in this economy: speculators and liquidity traders. Both types of agents are risk neutral and, for simplicity, there is no discounting. First, there is a large number of speculators, a continuum, who can, in the beginning of each period t , observe a common signal z_t on period t return, δ_t , by exerting effort $e > 0$. I assume that e is small enough so that, in equilibrium, some speculators always choose to become informed. I denote the measure of informed speculators by n_t . I also refer to these traders as "informed traders."

I assume that $z_t \in \{\delta_t, 0\}$ and that the probability that z_t perfectly reveals δ_t , $\Pr\{z_t = \delta_t\}$, depends on the availability of private information. This, in turn, depends on the activity chosen by the firm. Activity A is relatively easy to monitor, whereas activity B is more difficult: i) when the firm is engaged in activity A , the availability of private information is high, and $\Pr\{z_t = \delta_t\} = \bar{\alpha} < 1$; and ii) when the firm is engaged in activity B , the availability of private information is low, and $\Pr\{z_t = \delta_t\} = \underline{\alpha}$, where $0 < \underline{\alpha} < \bar{\alpha}$. For simplicity, I assume that the firm's activity is not directly observable to any trader.

In addition to speculators, there is in each period t a measure m of liquidity traders, indexed in the interval $[0, m]$, on both sides of the market, i.e., with either a need to sell or buy the asset. I take the trading motives of these liquidity traders as exogenously given and simply assume that each such trader i has utility c_i from buying or selling one unit of the risky asset in that period. I assume that c_i is independent across traders, and uniformly distributed between zero and one.^{2,3}

²It is common in the market microstructure literature to assume the existence of some liquidity traders with exogenous, non-speculative trading motives. The reason is that in the absence of any non-speculative trading motives there is, in equilibrium, no trading at all (see e.g., Milgrom and Stokey (1982)). I have chosen not to model the trading needs of the liquidity traders explicitly. To minimize their impact on the volume and volatility dynamics, I assume, however, that the trading needs of the liquidity traders are constant over time.

³I therefore assume that liquidity traders have no discretion in timing their trades, i.e., that they receive no utility from trading in, for instance, period $t+1$. Introducing limited discretion in timing for

Since I am also interested in the limit order placement strategies of traders, I assume that the equity markets are organized as a limit order market. The markets operate in the following manner: in each period, the opening price is set by the exchange at the expected value of the asset, conditional on all public information. After this, different traders can submit limit orders to the market. A computer, acting as a marketplace, first executes all limit orders that can be executed at the opening price. After this, in case of excess demand (supply) at the opening price, the computer increases (decreases) the market price to the level of the lowest unexecuted asks (highest unexecuted bids), and executes all limit orders that can be executed at that new price. The process continues until no limit orders cross. The price that prevails after this process becomes the closing price of the trading period. When there is excess demand at any given price, e.g., y buy orders and k sell orders, where $y > k$, I assume that each buyer receives stock *at that price* with an equal probability k/y . The order size is exogenously set as one unit per trader.

Note that the assumed market clearing is different from that of a typical call market where all trading takes place at a single price. I try to imitate the outcome of a continuous electronic limit order market, where traders submit orders through intermediaries, and there is uncertainty over the exact time their orders reach the market. Since I collapse a continuous trading day into a single period, I do not assume a single market clearing price nor do I impose price priority, but assume that, at any given price, all crossing orders are equally likely to be executed. Figure 1 clarifies the operation of these markets.⁴

In each period $t \leq T - 1$, the timing of events is as follows.

- i) The state of the economy $s_t \in \{A, B\}$ is realized. Speculators choose whether to pay e and obtain access to the informative signal z_t .
- ii) The periodic innovation $\delta_t \in \{-1, 1\}$ is realized. The informed traders observe a signal $z_t \in \{\delta_t, 0\}$.
- iii) The exchange announces an opening price equal to the expected value of the asset, conditional on all public information.
- iv) Traders submit limit orders (either one bid, one ask, or both) to the market, and the market clears according to the rules outlined above. All traders observe the entire sequence of prices and associated volumes of trade.
- v) A public signal $a_t = \delta_t$ is released with probability $\rho \geq 0$.

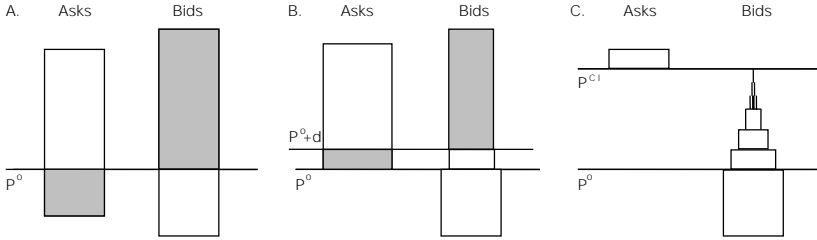
In period T , the timing of events is similar except that, after trading, there is a public signal (a quarterly report) that reveals all δ_τ for $\tau \leq T$. After this, the firm pays $F_T = \bar{F} + \sum_{t=1}^T \delta_t$ as a dividend to all shareholders.

Before proceeding, I introduce some additional notation: denote by P_t^o and P_t^{cl} the opening and closing prices of period $t > 0$. Whenever I write P_t , i.e., without any superscript, I am referring to the opening price. Denote by Ψ_t the

liquidity traders across days could lead to clustering of trading on some days, as described in Foster and Viswanathan (1990).

⁴For the informed traders to be able to make positive profits, it is crucial that the traders cannot condition on the contemporaneous order flow. However, in my setting, it would make little difference if traders were allowed to condition on contemporaneous trading volume. The issue of allowing traders to condition on contemporaneous trading volume is further discussed in Blume et al. (1994).

FIGURE 1
The Trading Mechanism



The width of the bars reflects the amount of limit orders at any given price. In Figure 1, this is initially constant for the range of bids and asks that are submitted to the market.

In Figure 1.A, the shaded areas show the quantities of crossing orders. These orders are matched, and trading takes place at the opening price P^0 . When there is excess demand at price P^0 , for instance y buy orders and k sell orders where $y > k$, we assume that each buyer receives stock at price P^0 with an equal probability k/y . In Figure 1.B, the bars now show the distributions of bids and asks after matching the crossing orders at price P^0 . As there is excess demand at price P^0 , the price rises to $P^0 + d$. The shaded areas now show the crossing orders at price $P^0 + d$. These orders are now matched, and trading takes place at price $P^0 + d$. In Figure 1.C, this process continues until there are no crossing orders. The price that prevails at the end of this process, P^{cl} , becomes the closing price of the period.

public information set at the beginning of period t . The public information set Ψ_t contains the price and volume for all transactions executed in the previous periods as well as all public announcements. In particular, Ψ_t contains all the aggregate daily volumes of trade up to period $t - 1$, $\{\omega_\tau\}_{\tau=1}^{t-1}$. Denote by R_t the traders' estimate of the probability of state A in period t , given Ψ_t , i.e., $R_t \equiv E[s_t = A \mid \Psi_t]$. Similarly, denote by α_t the probability that the informed traders receive a signal $z_t = \delta_t$ in period t , given Ψ_t , i.e., let $\alpha_t = \underline{\alpha} + R_t[\bar{\alpha} - \underline{\alpha}]$. Finally, let $F_t \equiv E[F_T \mid \Psi_t]$ and $\hat{\delta}_t \equiv E[\delta_t \mid \Psi_{t+1}]$.

To determine uniquely the trading volume by preventing equilibria where informed traders trade among themselves at the expected value of the asset, I assume, in addition, that each trader must pay a positive transaction cost h , where h is arbitrarily close to zero each time a transaction is completed. For the sake of exposition, because the transaction cost h is arbitrarily small, I will ignore this parameter except in the proofs of the propositions.

The objective of all traders is to maximize their utility, which is linear in their period T wealth and their utility from satisfying their liquidity needs. The information set of a speculator i , in the beginning of period t , Ω_t^i , contains Ψ_t as well as all past signals observed by trader i , $\{z_\tau^i\}_{\tau=1}^{t-1}$, where $z_\tau^i = z_\tau$ if trader i was informed in period τ and $z_\tau^i = \emptyset$ otherwise. That is, $\Omega_t^i = \{\Psi_t, \{z_\tau^i\}_{\tau=1}^{t-1}\}$. The existence of this past private information makes the model potentially very complicated. In this paper, however, I focus only on equilibria where z_t is perfectly revealed at the end of the period t and, therefore in equilibrium, the payoff relevant information contained in Ω_t^i is also contained in Ψ_t . Denote by $q_t^{\text{ask}}(P, a)$ and $q_t^{\text{bid}}(P, b)$ the unconditional probabilities that an ask a or a bid b ,

respectively, is executed at a price P in period t . Under the assumption that z_t is perfectly revealed in equilibrium, the maximization problem for an informed trader i in period t , after observing a private signal z_t , can now be written as,

$$(1) \quad \max_{b,a} E \left[[F_T - P] q_t^{\text{bid}}(P, b) + [P - F_T] q_t^{\text{ask}}(P, a) \mid \Psi_t, z_t \right].$$

A risk-neutral liquidity trader, on the other hand, maximizes the sum of his period t trading profit and his utility from trading. Therefore, the maximization problem for a liquidity trader i with a utility c_i from selling the asset is

$$(2a) \quad \max_{b,a} E \left[[F_T - P] q_t^{\text{bid}}(P, b) + [P - F_T + c_i] q_t^{\text{ask}}(P, a) \mid \Psi_t \right],$$

and the maximization problem for a liquidity trader j with a utility c_j from buying the asset is

$$(2b) \quad \max_{b,a} E \left[[F_T - P + c_j] q_t^{\text{bid}}(P, b) + [P - F_T] q_t^{\text{ask}}(P, a) \mid \Psi_t \right].$$

An equilibrium exists when informed speculators maximize (1), the liquidity traders maximize (2), uninformed speculators choose not to trade, z_t is perfectly revealed through traders' actions, and the number of informed traders is such that

$$(3) \quad E \left[\max_{b,a} E \left[[F_T - P] q_t^{\text{bid}}(P, b) + [P - F_T] q_t^{\text{ask}}(P, a) \mid \Psi_t, z_t \right] \mid \Psi_t \right] = e.$$

Equation (3) requires that the expected profits to informed traders be equal to their cost of information acquisition, e .⁵

Since the setup is symmetric in terms of magnitudes and probabilities of price increases and decreases, I will focus attention on symmetric Nash equilibria. I now proceed to the analysis of the equilibrium of the trading game for the risky asset.

III. Equilibrium

Proposition 1. The following is a Nash equilibrium of the trading game: In period t , a measure n_t^* of speculators obtain access to a private signal z_t . After the exchange sets an opening price $P_t^o = F_t$, these informed traders submit a bid and an ask with a limit price equal to $F_t + z_t$. A fraction c_t^* of liquidity traders with selling (buying) needs, where $1 > c_t^* > 0$, trade conservatively, and set their asks at $F_t + 1$ (bids at $F_t - 1$). The remaining fraction $(1 - c_t^*)$ of liquidity traders with selling (buying) needs trade aggressively, and set their asks at $F_t - 1$ (bids at $F_t + 1$). The closing price of period t is always $F_t + z_t$. The fraction of conservatively trading liquidity traders, c_t^* , is uniformly increasing in α_t . When e is small enough, the number of informed traders, n_t^* , is initially increasing, but later decreasing in α_t .

The proof is given in the Appendix.⁶ The intuition for why traders submit bids and asks with only two limit prices is as follows. Given such behavior by

⁵The maximization problem for uninformed speculators is similar to equation (2) with $c_i = 0$.

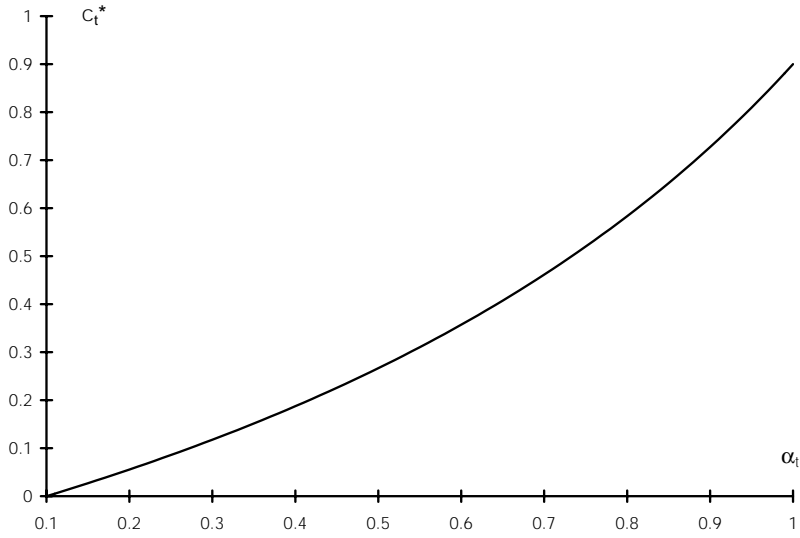
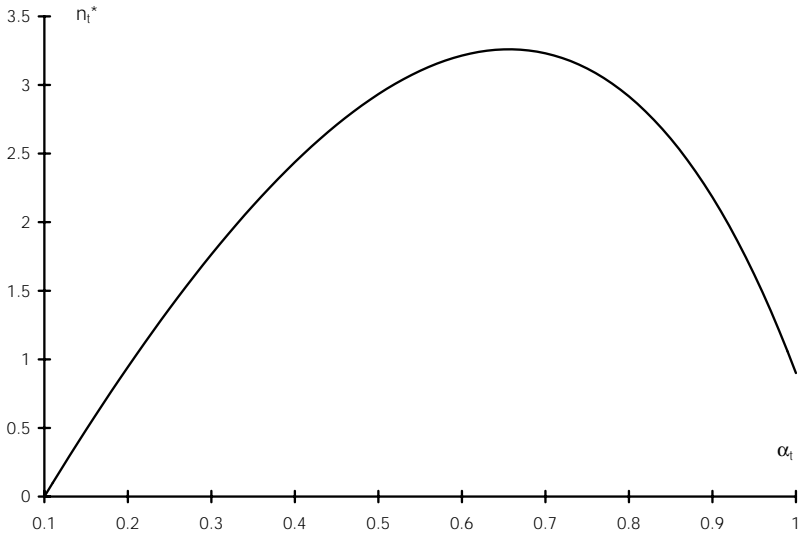
⁶Although I have not been able to show it formally, I suspect that this equilibrium is the unique symmetric equilibrium, and it may well be the unique equilibrium overall, apart from the ones where all liquidity traders post conservative limit orders or submit no orders at all.

the informed traders and other liquidity traders, a liquidity trader with a need to sell may do one of two things: first, he may set the limit price of his sell order conservatively, at $F_t + 1$, and sell only when the informed traders observe $z_t = 1$. Second, he may set his limit price aggressively, lower or equal to F_t , in order to have the possibility of trading with liquidity traders who have opposite liquidity needs in the event that $z_t \leq 0$. In a symmetric equilibrium such as this, the measure of liquidity traders with selling needs, who set their asks below F_t , is the same as that of liquidity traders with buying needs, who set their bids above F_t . Given this, the first of these should set their ask prices at $F_t - 1$ because the only instant in which they may fail to trade, in equilibrium, at price F_t is when there are informed traders who have observed $z_t = -1$. In this case, the value of the asset is $F_t - 1$ and the liquidity traders with selling needs are better off trying to sell to those traders who have set their bids at $F_t - 1$. The logic is similar for liquidity traders with buying needs. In equilibrium, those liquidity traders with small gains from trading submit conservative orders, whereas those liquidity traders with large gains from trading submit aggressive ones.

Because each informed trader is small, he is best off by trading on his private information, that is, by trying to buy when $z_t > 0$ and sell when $z_t < 0$. Given the behavior of the other traders, sending a bid and an ask with a limit price $F_t + z_t$ maximizes his expected profits. Note that given these strategies, the closing price of period t , P_t^{cl} , is always equal to $F_t + z_t$. Thus, the private information of the informed traders, z_t , is always perfectly revealed at the end of the period. In this model, competition among informed traders leads them to trade aggressively whenever they have (non-zero) private information and this reveals their private information to the market. Even though the informed traders' private information is perfectly revealed to the markets through this process, the expected profits to informed traders are positive. The profits occur since with some probability, an informed trader's order is executed at the opening price, F_t , at which initial trading takes place and this price does not reflect the period t private information, z_t . Note also that because z_t is always perfectly revealed, the next period's opening price, F_{t+1} , is equal to $P_t^{cl} = F_t + z_t$ when there is no public announcement and to $F_t + a_t$ otherwise.

It is intuitive that the proportion of liquidity traders who choose to trade conservatively, c_t^* , is increasing in the probability of the existence of private information in period t , α_t . The higher α_t is, the more severe the adverse selection problem faced by liquidity traders. For instance, if a liquidity trader sets an ask below the opening price F_t , it is always executed at F_t when δ_t is positive, but it is executed at that price only with some probability, which is decreasing in α_t when δ_t is negative. Similarly, as the probability of the existence of private information, α_t , increases, initially more speculators choose to search for private information. This, however, further worsens the adverse selection problem of the liquidity traders, and thus increases c_t^* . At some point in time, the adverse selection problem of the liquidity traders becomes so severe that only a few of them choose to submit aggressive limit orders. This reduces the expected profits to informed traders, and can lead to a decline in the number of speculators that acquire information. Figure 2 shows, for a parametric example, how c_t^* and n_t^* evolve.

FIGURE 2

 c_t^* and n_t^* as Functions of α_t when $m = 1$ and $e = 0.1$ Figure 2A. c_t^* as Function of α_t Figure 2B. n_t^* as Function of α_t 

If I interpret the aggressive limit orders as market orders, Proposition 1 shows that the proportion of limit orders by liquidity traders is increasing in the probability of private information, α_t . Below I show that market volatility is also increasing in α_t , implying that, in this model, the amount of limit orders by liquidity traders is higher in more volatile markets.

The equilibrium amount of limit orders by liquidity traders and the number of informed traders depend upon a single variable, α_t , which was defined as $\alpha_t = \Pr[z_t = \delta_t \mid \Psi_t]$. Remember also that $\alpha_t = \underline{\alpha} + R_t[\bar{\alpha} - \underline{\alpha}]$, where R_t is the probability of state A, given Ψ_t . This means that the evolution of the number of informed traders and the trading strategies of the liquidity traders actually depend upon the evolution of R_t . I now look at the evolution of R_t more closely. Denote by \hat{R}_t the following conditional probability, $\hat{R}_t = \Pr [s_t = A \mid \Psi_{t+1}]$, and denote by ω_t the period t trading volume. Note that $\omega_t = m(1 - c_t^*)$ when there is no private signal on δ_t , $z_t = 0$, and it is something greater than this, due to informed trading, in the event that there is, $z_t = \delta_t$.⁷ An application of Bayes rule gives

$$(4) \quad \hat{R}_t = \begin{cases} \frac{\bar{\alpha}R_t}{\alpha_t} & \text{if } \omega_t > m(1 - c_t^*) \\ \frac{(1 - \bar{\alpha})R_t}{(1 - \alpha_t)} & \text{if } \omega_t = m(1 - c_t^*). \end{cases}$$

Now, given the transition probabilities μ and ε between states A and B,

$$(5) \quad R_{t+1} = \hat{R}_t(1 - \mu) + (1 - \hat{R}_t)\varepsilon = \hat{R}_t(1 - \varepsilon - \mu) + \varepsilon.$$

These equations imply, for all observed values of R_{t+1} , that R_{t+1} , the probability of state A in period $t + 1$, given Ψ_{t+1} , is higher than R_t if there is evidence of informed trading in period t , and is lower than R_t otherwise. The result is due to the fact that informed trading is more likely to occur in the event that the firm is engaged in activity A as opposed to activity B.

Note that traders must use the volume of trade to correctly update their estimate of the state of the economy, which is the firm's activity or, equivalently, the probability of private information. The same information is not contained in the returns because the returns can also be non-zero due to a public signal. The trading volume helps separate private information arrivals from public information arrivals and, given that there are autocorrelated changes in the probability of private information over time, the trading volume contains useful information for the different groups of traders beyond that contained in the returns of the risky asset. Both speculators and liquidity traders need this information to follow their optimal strategies.

The probability of private information, α_t , affects the expected trading volume in three different ways: first, it is the probability that there is informed trading, second, it affects the number of informed traders and, third, the proportion of limit vs. market orders (conservative vs. aggressive limit orders) by liquidity traders. Both the state of the economy and beliefs about it matter for trading volume. The combination of these different effects leads to interesting dynamics for the trading volume.

⁷Recall that m is the measure of liquidity traders with, for instance, a need to sell the asset in period t .

Note that, for simplicity, the model assumes that not even the informed traders observe the firm's activity. If they did, more information about the state of the economy could be revealed through their trading than above. The reason is that the volume of trade in period t would then, if $z_t \neq 0$, provide information on how many traders chose to search information in period t , which, in this case, would depend on the state of the economy in the previous period. Because the previously informed traders would know the previous state of the world, other things being equal, fewer of them would search for information if this state were B rather than A . My main results concerning the price variability and volume dynamics do not seem sensitive to this assumption. The evolution equation for conditional price variability, which I will derive shortly, would nevertheless look different under this alternative assumption.

IV. Time-Series Properties

It turns out that for this model, it is much easier to characterize the behavior of price variability when it is calculated using opening prices rather than closing prices. The reason is that the daily signal for period t , a_t , becomes reflected only in the opening price of the following morning. When price variability is calculated from closing prices, the public signal introduces some negative autocorrelation in price variability for the first lag: revelation of private information today reduces the information revealed by the public signal tomorrow. This effect is not present for the opening prices. The statistical properties of closing prices are identical to those of the opening prices when the probability of a public signal is small. The following two propositions characterize some of the main dynamic properties of the model.

Proposition 2. The asset prices follow a martingale with respect to Ψ_t , i.e.,

$$E[P_{t+1} \mid \Psi_t] = P_t.$$

Proposition 3. Let $0 < s < T - t$. The following results hold,

$$\begin{aligned} \text{cov} [(P_{t+1} - P_t)^2, \omega_t \mid \Psi_t] &> 0, \\ \text{cov} [(P_{t+1} - P_t)^2, \omega_t] &> 0, \\ \text{cov} [(P_{t+s+1} - P_{t+s})^2, (P_{t+1} - P_t)^2] &= \\ \left[\frac{\varepsilon \mu (1 - \rho)^2 [\bar{\alpha} - \alpha]^2}{(\varepsilon + \mu)^2} \right] (1 - \varepsilon - \mu)^s &> 0. \end{aligned}$$

The proofs of these and all the remaining propositions are given in the Appendix. The asset price follows a martingale with respect to Ψ_t and the price variability and trading volume are positively correlated both unconditionally and conditional on Ψ_t . In addition, the autocorrelation function for price variability is positive and geometrically decaying, very much like the autocorrelation functions for price variability of stocks and stock indices documented in the empirical research (see, e.g., Brock et al. (1996)).

The positive correlation between volume and price variability, conditional on the current information set, arises from the fact that the informed traders trade only when they receive (non-zero) private information, and that their trading carries information and affects prices. This effect is so strong that there is a positive correlation between volume and price variability, even when I am not conditioning on the public information set.

The evolution of price variability depends on the evolution of the availability of private information. When the firm is engaged in activity *A*, which is relatively easy to monitor, the speculators receive private information with a higher probability and, since the private information is revealed through their trading, prices are more volatile than when the firm is engaged in activity *B*. If the availability of private information changes according to a two-state Markov process, as I assume, the autocorrelation function for price variability is geometrically decaying. This occurs because the longer the time period between two return observations, the larger the probability that the firm has changed its activity during that time and that the availability of private information has thereby changed.

Interestingly, autocorrelation for volume is indeterminate in this model. If the number of speculators was constant and the liquidity traders did not change their strategies over time, the trading volume would be positively autocorrelated, similarly to the price variability. The endogenous behavior of liquidity traders introduces, however, negative autocorrelation in trading volume: liquidity traders react to a large trading volume by reducing the proportion of market orders. All else being equal, this reduces the expected trading volume in the next period. The behavior of speculators, on the other hand, can introduce either positive or negative autocorrelation in trading volume, depending on whether we are in the range of α_t where the number of informed speculators is increasing or decreasing in α_t (see Figure 2). Whether the autocorrelation in trading volume is positive or negative depends upon which of these three effects dominates.

It is equally interesting to characterize the dynamics of expected variance of price changes and trading volume. Propositions 4 and 5 show that the conditional variance in this model is positively autocorrelated and mean reverting, as is directly assumed in the GARCH literature. Before proceeding, I introduce the following notation. Denote by σ_t^2 the conditional variance, $\sigma_t^2 = \text{var}[P_{t+1} - P_t \mid \Psi_t]$, and let $E_t(\cdot)$ denote the expectation operator conditioned on the public information set Ψ_t . I can now state the following.

Proposition 4. $\text{cov}[\sigma_{t+s}^2, \sigma_t^2] > 0$.

Proposition 5. σ_t^2 is mean reverting. In particular, $E_t[\sigma_{t+1}^2] < \sigma_t^2$ when $\sigma_t^2 > \bar{\sigma}^2$ and $E_t[\sigma_{t+1}^2] > \sigma_t^2$ when $\sigma_t^2 < \bar{\sigma}^2$, where

$$\bar{\sigma}^2 = \rho + (1 - \rho) \left(\alpha + \frac{\varepsilon(\bar{\alpha} - \alpha)}{(\varepsilon + \mu)} \right).$$

The Appendix shows that $\sigma_t^2 = \rho + (1 - \rho)\alpha_t$. This result is very intuitive: ρ is the probability that δ_t is announced through a public announcement, and α_t the independent probability with which the informed traders observe δ_t , given Ψ_t . Given that the trading by the informed traders is fully revealing, the value of δ_t is revealed with probability $\rho + (1 - \rho)\alpha_t$. As the absolute value of the periodic

innovations is always equal to one, this is also the expected variance of the price changes. Since $\alpha_t = \underline{\alpha} + R_t[\bar{\alpha} - \underline{\alpha}]$, σ_t^2 can also be written as $\rho + (1 - \rho)[\underline{\alpha} + R_t[\bar{\alpha} - \underline{\alpha}]]$. The above two results, positive autocorrelation and mean reversion in conditional variance, follow from similar properties in R_t .

I am interested in characterizing more explicitly the evolution of the conditional variance. Given that the conditional variance is a function of R_t , and the updating rule for R_t depends on the current period's trading volume by equation (4), trading volume enters the evolution equation for conditional variance. Indeed, when $0 < \rho < 1$, price changes are not sufficient statistics for predicting changes in the conditional variance, but information on the volume of informed trading is needed. If this is not observable (to econometricians), one might use information on both total trading volume and price innovations to estimate it. For this model, I can characterize explicitly the transition equation for the conditional variance, and show its dependence on trading volume. To do this, some additional notation is needed: let $\underline{\sigma}^2 = \rho + (1 - \rho)\underline{\alpha}$.

Solving for the conditional variance process gives

$$(6) \quad \sigma_t^2 = \varphi + \zeta(\sigma_{t-1}^2) + \phi(\sigma_{t-1}^2) \left(\frac{\omega_{t-1} - m(1 - c_{t-1}^*)}{x_{t-1}^*} \right),$$

$$\text{where } x_{t-1}^* = \max \left\{ mc_{t-1}^*, \frac{m(1 - c_{t-1}^*)n_{t-1}}{m(1 - c_{t-1}^*) + n_{t-1}} \right\},$$

$$\varphi = \rho + (1 - \rho) [\underline{\alpha} + \varepsilon [\bar{\alpha} - \underline{\alpha}]],$$

$$\zeta = \frac{(1 - \varepsilon - \mu)(1 - \rho)(1 - \bar{\alpha}) [\sigma_{t-1}^2 - \underline{\sigma}^2]}{(1 - \sigma_{t-1}^2)}, \quad \text{and}$$

$$\phi = (1 - \varepsilon - \mu)(1 - \rho) [\sigma_{t-1}^2 - \underline{\sigma}^2] \left[\frac{\bar{\alpha}}{\sigma_{t-1}^2 - \rho} - \frac{(1 - \bar{\alpha})}{1 - \sigma_{t-1}^2} \right].$$

I can show that if the trading volume in period $t - 1$ is low, i.e., $\omega_{t-1} = m(1 - c_{t-1}^*)$, then $\sigma_t^2 < \sigma_{t-1}^2$, and if the trading volume is high, i.e., $\omega_{t-1} > m(1 - c_{t-1}^*)$, then $\sigma_t^2 > \sigma_{t-1}^2$. Thus, the conditional variance always increases if there is evidence of informed trading, and decreases otherwise. Note that in the special case where ρ , the probability of public information arrival, is zero, the above equation can be written as

$$\sigma_t^2 = \varphi + \zeta(\sigma_{t-1}^2) + \phi(\sigma_{t-1}^2) (P_t - P_{t-1})^2,$$

which is close to the evolution equation for variance typically assumed in the GARCH literature.⁸ In this model, when $1 > \rho > 0$, the volume of trade allows an econometrician to separate private information arrivals from public information arrivals and, given that there is autocorrelation in the private information arrival, to better estimate the conditional variance as compared to what would be

⁸In the GARCH literature, the conditional variance is typically assumed to evolve according to

$$\sigma_t^2 = \alpha + \beta\sigma_{t-1}^2 + \theta (P_t - P_{t-1})^2.$$

possible using data on prices alone. This result should continue to hold even in more general settings where there are changes also in the arrival rate of public information over time.

The final time-series result concerns the relationship between expected price variability and the expected trading volume, $\omega_t^e = E[\omega_t \mid \Psi_t]$.

Proposition 6. The conditional variance, σ_t^2 , and the conditional expected trading volume, ω_t^e , can be either positively or negatively correlated random variables.

The result follows from the fact that ω_t^e can be either an increasing or decreasing function of α_t , depending on the possible range of values for α_t , as well as the other parameters of the model. The result that the expected trading volume can decrease in the probability of private information is somewhat surprising. It does occur, however, because of the endogenous trading strategies of the liquidity traders. In fact, in the limiting case that $e = 0$, trading volume is m both when $\alpha_t = 0$, all liquidity traders trade aggressively and there are no informed traders, and when $\alpha_t = 1$, all liquidity traders trade conservatively, but information is always perfectly revealed. For intermediate values of α_t , the expected trading volume can be either higher or lower than m . Figure 3 shows the expected trading volume for a parametric example. So it is possible that volatile markets, with much private information, are characterized by either high or low trading volume. In both cases, however, the proportion of limit orders by uninformed traders, c_t^* , is high.⁹

V. Conclusion

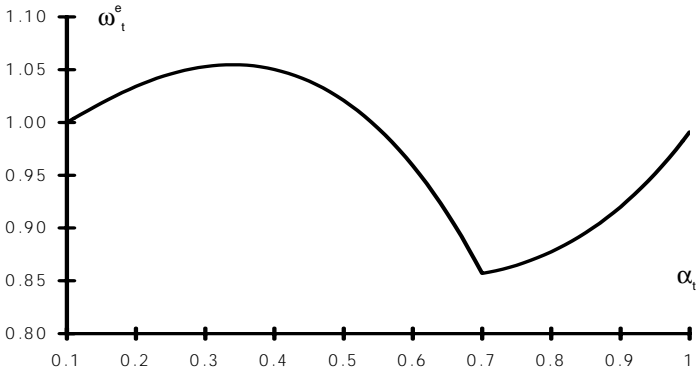
This paper studies an asset market where the availability of private information is stochastically changing over time due to changes in the source of uncertainty in the asset returns. In equilibrium, liquidity traders and speculators use past periods' trading volume to estimate the availability of private information. As the public estimate on the availability of private information increases, liquidity traders become wary and start posting more conservative limit orders. Initially, the number of informed traders increases but, in response to more conservative trading by liquidity traders, it may subsequently decrease.

My model is consistent with several empirically observed patterns. Because the trading by informed traders reveals private information, there is a positive correlation between price variability and trading volume. The price variability depends on the availability of private information and inherits any autocorrelation in the process that determines it. When the availability of private information

⁹One empirical observation that I did not address, the so-called leverage effect, is an observation that prices become more volatile after negative returns than after positive ones. In fact, the current model could be adjusted to account for this phenomenon as well if I were to assume that the two projects have different mean returns: for instance, assume that $\Pr[\delta_t = 1 \mid A] < 1/2$. However, with this assumption, the model becomes quite complicated. This empirical observation can be related to changes in the level of volatility of the underlying returns (not only the availability of private information). It may well be that high volatility periods, associated with economic or financial distress, do have lower mean returns on firms' investments. Alternatively, it may also be that the expected return on the risky asset depends on volatility, and this explains why high future volatility is associated with negative stock returns.

FIGURE 3

The Expected Trading Volume as a Function of α_t when $m = 1$ and $e = 0.1$



changes according to a two-state Markov process, as I assumed, the autocorrelation function for variance is positive and geometrically decaying. This is very much like the autocorrelation functions of individual stocks and stock indices estimated by the empirical research. I showed that the conditional variance is autocorrelated and mean reverting and that it may be either positively or negatively correlated with the expected trading volume. Finally, I showed that price changes are not sufficient statistics to characterize the evolution of conditional variance, but that information on trading volume is also needed.

The model has several empirical implications that could be tested in future research. For instance, the model provides a closed form solution for the conditional variance process. Second, it predicts that the conditional variance is positively correlated with measures of information asymmetry in the market, such as the bid-ask spread or, as predicted by this model, the proportion of limit orders by uninformed traders.

Appendix

Proof of Proposition 1. Given the behavior of the other informed traders and the liquidity traders, the incentives of an informed trader to follow the above strategy profile are satisfied: setting a bid and an ask at $E[F_T | \Psi_t, z_t] \pm h = F_t + z_t \pm h$, where h is the transaction cost, maximizes his expected trading profits. If $z_t = 1$, for instance, to maximize 1, an informed trader must set his ask at or above $F_t + z_t + h$ to avoid making losses. On the other hand, he profits from buying the asset at any transaction price strictly below $F_t + z_t - h$. Given my market clearing rule and the strategies of the other traders, he will make a trading profit if and only if he succeeds to trade at $P_t^o = F_t$. Therefore, his expected profits are the same for any bid at or above F_t including $F_t + z_t - h$. The situation is similar if $z_t = -1$. If $z_t = 0$, the informed trader does not expect to trade but is willing to sell if the

price is above $F_t + h$ and buy if the price is below $F_t - h$. As he expects the price to remain at F_t , setting a bid and an ask at these prices is optimal.¹⁰

Whether the high ask price of a liquidity trader with selling need is $F_t + 1 + h$ or $F_t + 1 - h$ depends on whether $mc_t^* > (m(1 - c_t^*)n_t^*) / (m(1 - c_t^*) + n_t^*)$. This inequality determines whether the number of liquidity traders selling at $F_t + 1$, i.e., mc_t^* , is greater than the number of liquidity traders buying at $F_t + 1$ when $z_t = 1$. The latter is the number of liquidity traders who cannot complete their orders at the opening price F_t , when $z_t = 1$. Whichever group is larger depends on the values of e and α_t . Let us here consider the case where the former group, i.e., mc_t^* , is larger. In this case, the conservatively trading liquidity traders set their ask prices at $F_t + 1 - h$ and bid prices at $F_t - 1 + h$. The second case is similar and therefore omitted.

Given the behavior of the other traders, a liquidity trader with a need to *sell* may do one of two things: first, he may set the limit price of his sell order at $F_t + 1 - h$ and sell only when informed traders observe $z_t = 1$, either to the informed traders or to liquidity traders with buying needs who fail to buy at a lower price (note that, if $n_t^* > mc_t^*$, as I will later show, he is guaranteed a sale at this price, so setting a lower limit price does not make sense). Or, second, he may set his limit price lower than F_t in order to have the possibility of trading even in the event that $z_t \leq 0$. Given that the number of liquidity traders with selling needs setting their asks below F_t is the same as the number of liquidity traders with buying needs setting their bids above F_t , the former group may actually set their asks at $F_{t-1} - 1 + h$, because the only instant that they may fail to trade in equilibrium at the opening price F_t is when there are informed traders who have observed $z_t = -1$. In this case, they are better off trying to trade with traders who have set their bids at $F_{t-1} - 1 + h$. The logic is similar for liquidity traders with large buying needs. Because of this, the liquidity traders, in equilibrium, use only two of all possible limit prices: $F_t + 1 - h$ and $F_{t-1} - 1 + h$. Note that, given the behavior of the informed traders, the closing price of the period is always $F_t + z_t(1 - h)$.

Let $\pi_a^H(c)$ be the sum of a liquidity trader's utility from satisfying his liquidity need when he obtains a utility c from selling the asset in period t and his expected period t trading profit from setting a *high ask* price $F_t + 1 - h$. Define π_a^L , π_b^H , and π_b^L accordingly. Given that $P_t^{cl} = E[F_T \mid \Psi_t, z_t]$, the period t trading profit to trader i , evaluated before the period t public announcement, can be written as $x_i[P_t^{cl} - P^i]$. Here, $x_i \in \{1, -1\}$ is the quantity traded and P^i is i 's transaction price. Equilibrium prevails when a measure mc_t^* of liquidity traders with selling (buying) needs set their asks (bids) at $F_t + 1 - h$ ($F_t - 1 + h$) and a measure $m(1 - c_t^*)$ of liquidity traders with selling (buying) needs set their asks (bids) at $F_t - 1 + h$ ($F_t + 1 - h$), and both $\pi_a^H(c_t^*) = \pi_a^L(c_t^*)$ and $\pi_b^L(c_t^*) = \pi_b^H(c_t^*)$. Straightfor-

¹⁰For a Nash equilibrium, it is sufficient that the informed traders weakly prefer their equilibrium strategies. To rule out "unlikely" equilibria in games where players in equilibrium are indifferent between various actions, game theorists have introduced equilibrium refinements that essentially require the strategies to be optimal even if there are small possibilities of independent "mistakes" by other players (in the jargon of game theory such equilibrium is called "trembling hand perfect equilibrium"). I can show that my equilibrium strategy for an informed trader is the only strategy that would survive such equilibrium refinement.

ward calculations give $\pi_a^H(c_i^*) = \pi_b^L(c_i^*) = (\alpha_t/2)(c_i^* - 2h)$ and $\pi_a^L(c_i^*) = \pi_b^H(c_i^*) = (c_i^* - h) - (\alpha_t/2)(n_t^*/(m(1 - c_i^*) + n_t^*))(1 - h)$. Setting $\pi_a^H(c_i^*) = \pi_a^L(c_i^*)$ gives

$$(7) \quad \left(1 - \frac{\alpha_t}{2}\right) c_i^* - \frac{\alpha_t}{2} \left(\frac{n_t^*}{m(1 - c_i^*) + n_t^*}\right) (1 - h) - (1 - \alpha_t)h = 0.$$

An uninformed speculator's problem is identical to that of a liquidity trader whose $c_i = 0$. If he submitted orders, they would therefore be conservative and his expected profits would equal zero. Therefore, an uninformed speculator is willing to stay out of the market.

An overall equilibrium exists when c_i^* satisfies equation (7) and n_t^* is such that equation (3), which requires that the expected profits to informed traders are equal to their cost of information acquisition, holds, i.e.,

$$(8) \quad \alpha_t \left(\frac{m(1 - c_i^*)}{m(1 - c_i^*) + n_t^*}\right) (1 - h) = e.$$

Recall that I have ruled out the possibility that $n_t^* = 0$ by assuming that e is small enough so that some speculators always choose to become informed. A sufficient assumption for this is that $e \leq \underline{\alpha}$.

Ignoring h , equations (7) and (8) imply

$$\begin{aligned} n_t^* &= m \left(\frac{2\alpha_t + 3\alpha_t e - 2\alpha_t^2 - 2e - e^2}{(2 - \alpha_t)e}\right) \\ \text{and } c_i^* &= \frac{\alpha_t - e}{2 - \alpha_t}. \end{aligned}$$

I can now verify that c_i^* is increasing in α_t , and that $n_t^* > mc_i^*$. The second derivative of n_t^* with respect to α_t is equal to $-2m(2 - e)^2/(e(2 - \alpha_t)^3)$, which is negative for all possible values for α_t . The first derivative of n_t^* with respect to α_t is initially positive, but reaches zero at $\alpha_t = 2 - \sqrt{2}(1 - e/2) > 0$. When e is less than $2 - 2/\sqrt{2}$, this critical value for α_t is less than one. In this case, n_t^* is decreasing in α_t for large values of α_t . \square

I now derive the equation for the expected trading volume: when $z_t = 0$, the trading volume is $m(1 - c_i^*)$ as the aggressive orders of liquidity traders are executed at price F_t and there is no further trading. When $z_t \neq 0$, the trading volume is $m(1 - c_i^*)$ at the opening price F_t , after which it is the maximum of mc_i^* and $(m(1 - c_i^*)n_t^*)/(m(1 - c_i^*) + n_t^*)$ at price $F_t + z_t$. Assume, for instance, that $z_t = 1$. mc_i^* is the number of liquidity traders with selling needs who have set their limit price at $F_t + 1$ ($\pm h$) and $(m(1 - c_i^*)n_t^*)/(m(1 - c_i^*) + n_t^*)$ is the number of liquidity traders with large buying needs, who could not complete their orders at price F_t . If $(m(1 - c_i^*)n_t^*)/(m(1 - c_i^*) + n_t^*) \geq mc_i^*$, it is optimal for the liquidity traders with small selling needs to set their asks at $F_t + 1 + h$ as they can be assured of a sale even at this price, in the event that $z_t = 1$. If $mc_i^* > (m(1 - c_i^*)n_t^*)/(m(1 - c_i^*) + n_t^*)$, on the other hand, competition leads them to set their ask prices at $F_t + 1 - h$ in order to guarantee a trade. In both cases, the speculators are willing to be counterparts to the excess demand (or supply) from these traders, of size $|mc_i^* - (m(1 - c_i^*)n_t^*)/(m(1 - c_i^*) + n_t^*)|$, at

this price. The total trading volume is then $m(1 - c_t^*) + x_t^*$ when $z_t \neq 0$, where $x_t^* = \max\{mc_t^*, (m(1 - c_t^*)n_t^*) / (m(1 - c_t^*) + n_t^*)\}$ and it is $m(1 - c_t^*)$ when $z_t = 0$. The expected trading volume, conditional on Ψ_t , is therefore,

$$\begin{aligned}\omega_t^e &= E[\omega_t | \Psi_t] \\ &= m(1 - c_t^*) + \alpha_t x_t^*.\end{aligned}$$

Proof of Proposition 2.

$$\begin{aligned}E[P_{t+1} | \Psi_t] &= \bar{F} + \sum_{\tau=0}^{t-1} \hat{\delta}_\tau + E[E[\delta_t | \Psi_{t+1}] | \Psi_t] \\ &= \bar{F} + \sum_{\tau=0}^{t-1} \hat{\delta}_\tau + \frac{\rho + (1 - \rho)\alpha_t}{2} - \frac{\rho + (1 - \rho)\alpha_t}{2} \\ &= \bar{F} + \sum_{\tau=0}^{t-1} \hat{\delta}_\tau = P_t. \quad \square\end{aligned}$$

Proof of Proposition 3.

Statement 1:

$$\begin{aligned}\text{cov}[(P_{t+1} - P_t)^2, \omega_t | \Psi_t] &= E[(P_{t+1} - P_t)^2 \omega_t | \Psi_t] - E[(P_{t+1} - P_t)^2 | \Psi_t] E[\omega_t | \Psi_t] \\ &= \alpha_t (m(1 - c_t^*) + x_t^*) + (1 - \alpha_t)\rho m(1 - c_t^*) \\ &\quad - (\rho + (1 - \rho)\alpha_t) [m(1 - c_t^*) + \alpha_t x_t^*] \\ &= \alpha_t x_t^* (1 - \alpha_t) (1 - \rho) > 0.\end{aligned}$$

Statement 2:

$$\begin{aligned}\text{cov}[(P_{t+1} - P_t)^2, \omega_t] &= E[(P_{t+1} - P_t)^2 \omega_t] - E(P_{t+1} - P_t)^2 E\omega_t \\ &= E\left[E[(P_{t+1} - P_t)^2 \omega_t | \Psi_t]\right] \\ &\quad - E\left[E[(P_{t+1} - P_t)^2 | \Psi_t]\right] E[E[\omega_t | \Psi_t]] \\ &= E[\alpha_t (m(1 - c_t^*) + x_t^*) + (1 - \alpha_t)\rho m(1 - c_t^*)] \\ &\quad - E[(\rho + (1 - \rho)\alpha_t)] E[m(1 - c_t^*) + \alpha_t x_t^*] \\ &= (1 - \rho) [E(\alpha_t x_t^*) (1 - E\alpha_t) + mE\alpha_t E c_t^* - mE(\alpha_t c_t^*)] \\ &\geq (1 - \rho) [mE(\alpha_t c_t^*) (1 - E\alpha_t) + mE\alpha_t E c_t^* - mE(\alpha_t c_t^*)] \\ &= (1 - \rho) E\alpha_t m [E c_t^* - E(\alpha_t c_t^*)] > 0.\end{aligned}$$

Statement 3: Introducing a random variable S_t such that $S_t = 1$ when $s_t = A$ and $S_t = 0$ when $s_t = B$, and using the law of iterated expectations,

$$\begin{aligned}E[S_{t+1} | S_t] &= S_t(1 - \varepsilon - \mu) + \varepsilon \\ E[S_{t+2} | S_t] &= E[E[S_{t+2} | S_{t+1}] | S_t] = E[[S_{t+1}(1 - \varepsilon - \mu) + \varepsilon] | S_t] \\ &= S_t(1 - \varepsilon - \mu)^2 + \varepsilon(1 - \varepsilon - \mu) + \varepsilon,\end{aligned}$$

and, more generally, that

$$\begin{aligned} E[S_{t+s} | S_t] &= S_t(1 - \varepsilon - \mu)^s + \sum_{i=0}^{s-1} \varepsilon(1 - \varepsilon - \mu)^i \\ &= S_t(1 - \varepsilon - \mu)^s + \frac{\varepsilon - \varepsilon(1 - \varepsilon - \mu)^s}{\varepsilon + \mu}. \end{aligned}$$

Using this result and the fact that $ES_t = ES_t^2 = \varepsilon/(\varepsilon + \mu)$, for all t ,

$$\begin{aligned} &\text{cov} \left[(P_{t+s+1} - P_{t+s})^2, (P_{t+1} - P_t)^2 \right] \\ &= \Pr \left[\widehat{\delta}_{t+s} \neq 0 \text{ and } \widehat{\delta}_t \neq 0 \right] - \Pr \left[\widehat{\delta}_{t+s} \neq 0 \right] \Pr \left[\widehat{\delta}_t \neq 0 \right] \\ &= E \left[E \left[(\rho + (1 - \rho) [\underline{\alpha} + S_{t+s}(\overline{\alpha} - \underline{\alpha})]) (\rho + (1 - \rho) [\underline{\alpha} + S_t(\overline{\alpha} - \underline{\alpha})]) \mid S_t \right] \right. \\ &\quad \left. - [(\rho + (1 - \rho) E[\underline{\alpha} + S_t(\overline{\alpha} - \underline{\alpha})])]^2 \right] \\ &= E \left(\rho + (1 - \rho) \left[\underline{\alpha} + \left[S_t(1 - \varepsilon - \mu)^s + \frac{\varepsilon - \varepsilon(1 - \varepsilon - \mu)^s}{\varepsilon + \mu} \right] (\overline{\alpha} - \underline{\alpha}) \right] \right) \\ &\quad \cdot (\rho + (1 - \rho) [\underline{\alpha} + S_t(\overline{\alpha} - \underline{\alpha})]) - \left[\left(\rho + (1 - \rho) \left[\underline{\alpha} + \frac{\varepsilon}{\varepsilon + \mu} (\overline{\alpha} - \underline{\alpha}) \right] \right) \right]^2 \\ &= (1 - \rho)^2 (\overline{\alpha} - \underline{\alpha})^2 \left[E \left[S_t^2 (1 - \varepsilon - \mu)^s + S_t \frac{\varepsilon - \varepsilon(1 - \varepsilon - \mu)^s}{\varepsilon + \mu} \right] - \left(\frac{\varepsilon}{\varepsilon + \mu} \right)^2 \right] \\ &= \frac{\varepsilon \mu (1 - \rho)^2 (\overline{\alpha} - \underline{\alpha})^2}{(\varepsilon + \mu)^2} (1 - \varepsilon - \mu)^s. \quad \square \end{aligned}$$

Derivation of the conditional variance. First, note that $P_{t+1} - P_t = \widehat{\delta}_t \equiv E[\delta_t | \Psi_{t+1}]$. The martingale property of prices, formally shown above, implies that $E[P_{t+1} - P_t | \Psi_t] = 0$. Given this, and the fact that $\widehat{\delta}_t$ is either zero, one, or minus one, I obtain that $\text{var}[P_{t+1} - P_t | \Psi_t] = E[\widehat{\delta}_t^2 | \Psi_t]$. Straightforward calculations give

$$\sigma_t^2 = \text{var}[P_{t+1} - P_t | \Psi_t] = \rho + (1 - \rho)\alpha_t.$$

Proof of Proposition 4. According to Bayes rule R_t , the probability of state A given Ψ_t , evolves according to equations (4) and (5). Now, first note that $E_t[\widehat{R}_t] = R_t$, and that

$$\begin{aligned} E_t[R_{t+1}] &= E[R_{t+1} | R_t] = E[\widehat{R}_t](1 - \varepsilon - \mu) + \varepsilon \\ &= R_t(1 - \varepsilon - \mu) + \varepsilon. \end{aligned}$$

Similarly, as in the proof of Proposition 3, I can use the law of iterated expectations to obtain

$$\begin{aligned} E[R_{t+s} | R_t] &= R_t(1 - \varepsilon - \mu)^s + \sum_{i=0}^{s-1} \varepsilon(1 - \varepsilon - \mu)^i \\ &= R_t(1 - \varepsilon - \mu)^s + \frac{\varepsilon - \varepsilon(1 - \varepsilon - \mu)^s}{\varepsilon + \mu}. \end{aligned}$$

Using the law of iterated expectations and Jensen's inequality,

$$\begin{aligned}
 \text{cov} [\sigma_{t+s}^2, \sigma_t^2] &= E([\sigma_{t+s}^2 - E\sigma_{t+s}^2][\sigma_t^2 - E\sigma_t^2]) \\
 &= (1 - \rho)^2 E_{R_t} E[[\alpha_{t+s} - E\alpha_{t+s}][\alpha_t - E\alpha_t] | R_t] \\
 &= (1 - \rho)^2 E_{R_t} E\left[\left[\alpha_{t+s} - \left(\frac{\varepsilon}{\varepsilon + \mu}[\bar{\alpha} - \underline{\alpha}] + \underline{\alpha}\right)\right]\right. \\
 &\quad \cdot \left.\left[\alpha_t - \left(\frac{\varepsilon}{\varepsilon + \mu}[\bar{\alpha} - \underline{\alpha}] + \underline{\alpha}\right)\right] \mid R_t\right] \\
 &= (1 - \rho)^2 (\bar{\alpha} - \underline{\alpha})^2 \\
 &\quad \cdot E_{R_t} E\left[\left(R_{t+s} - \frac{\varepsilon}{\varepsilon + \mu}\right)\left(R_t - \frac{\varepsilon}{\varepsilon + \mu}\right) \mid R_t\right] \\
 &= (1 - \rho)^2 (\bar{\alpha} - \underline{\alpha})^2 \\
 &\quad \cdot E_{R_t} \left[\left(R_t(1 - \varepsilon - \mu)^s - \frac{\varepsilon(1 - \varepsilon - \mu)^s}{\varepsilon + \mu}\right)\left(R_t - \frac{\varepsilon}{\varepsilon + \mu}\right)\right] \\
 &> (1 - \rho)^2 (\bar{\alpha} - \underline{\alpha})^2 \left(E[R_t](1 - \varepsilon - \mu)^s - \frac{\varepsilon(1 - \varepsilon - \mu)^s}{\varepsilon + \mu}\right) \\
 &\quad \cdot \left(E[R_t] - \frac{\varepsilon}{\varepsilon + \mu}\right) = 0. \quad \square
 \end{aligned}$$

Proofs of Propositions 5 and 6. The mean reversion follows directly from the definition of σ_t^2 and the properties of R_t , stated above. To prove Proposition 6, I must show that both positive and negative correlations are possible. I will use the fact that two increasing functions of the same random variable are positively correlated; and the fact that one decreasing and one increasing function of the same random variable are negatively correlated. To prove the first claim (the proof of the second is similar), let $\Psi(x)$ and $\Phi(x)$ be strictly increasing functions. Consider two i.i.d. random variables x and y . Then

$$\begin{aligned}
 E[\Psi(x) - \Psi(y)][\Phi(x) - \Phi(y)] &> 0 \\
 \Leftrightarrow E[\Psi(x)\Phi(x)] - E\Psi(x)E\Phi(y) - E\Psi(y)E\Phi(x) + E\Psi(y)E\Phi(y) \\
 &= 2E[\Psi(x)\Phi(x)] - 2E\Psi(x)E\Phi(x) = 2\text{cov}[\Psi, \Phi] > 0.
 \end{aligned}$$

Now consider the parametric example in Figure 3. If $\underline{\alpha} = 0.7$, both the expected price variability and trading volume are increasing functions of α_t and thus positively correlated. On the contrary, if $\underline{\alpha} = 0.4$ and $\bar{\alpha} = 0.6$, the expected trading volume is decreasing in α_t , and the two variables are negatively correlated. \square

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