Simulation of non-Hermitian quantum mechanics with a superconducting quantum processor

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Hermitian vs non-Hermitian quantum mechanics

Standard Hermitian quantum mechanics:

- Observables are Hermitian operators.
- Unitary dynamics- generated by Hermitian generators.
- Real eigenvalue spectrum $\Rightarrow$ physically realizable eigenstates.

[1] *Phys. Rev. Lett.* **80**, 5243–5246 (1998).

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Q: Whether a non-Hermitian operators can be an observable?

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Q: Whether a non-Hermitian operators can be an observable?
A: Yes, indeed!
Example– non-Hermitian Hamiltonian operators which satisfy $\mathcal{P}$(parity) $\mathcal{T}$(time)-symmetry also possess real eigenvalues[1], i.e., $[H, \mathcal{PT}] = 0$, where, $\mathcal{P}{=}\sigma_x$ and $\mathcal{T}$ is the complex conjugation.

Non-Hermitian operators can also have real expectation values and thus can be realized as physical observables.

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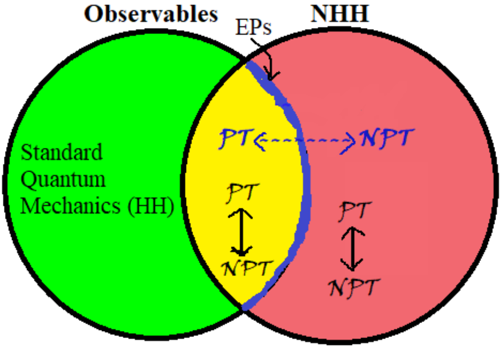
Illustration of operator space based on Hermiticity



- The yelloe region represents Non-Hermitian Hamiltonians (NHH) with real eigenvalues. Our interest lies in this yellow region and the blue EP boundary.
- Characterization of single-qutrit non-Hermitian dynamics.
- Observing the effect of NH dynamics on quantum correlations.

Hamiltonian

$$H_{\mathrm{q}} = \sigma_x + ir\sigma_z, \tag{1}$$

- r is a real parameter.
- The eigenvalues are $\pm\sqrt{1-r^2}$.
- The condition for non-Hermiticity is simply $r \neq 0$.
- Non-Hermitian regime of H_q can have following classifications [2]

$\mathcal{PT}$ symmetric

- $|r| < 1$
- Real eigenvalues
- Eigenvectors: $|\psi_\pm\rangle$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} ir \pm \sqrt{r^2-1} \\ 1 \end{pmatrix}$$

Exceptional point

- $|r| = 1$
- Eigenvalues=0
- Eigenvector:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

Broken $\mathcal{PT}$-symmetry

- $|r| > 1$
- Imag. eigenvalues
- Eigenvectors: $|\psi_\pm\rangle$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} ir \pm i\sqrt{r^2-1} \\ 1 \end{pmatrix}$$

[2] Science, **364**, 878-880 (2019).

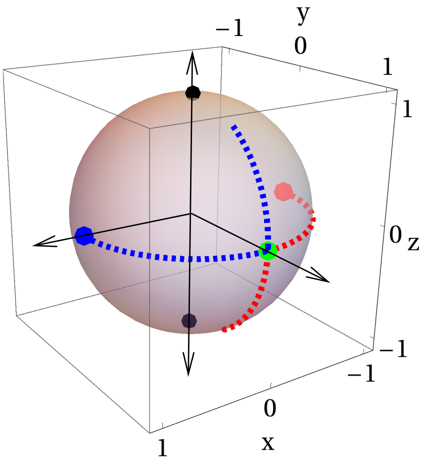
non-Hermitian Hamiltonian

$$H_{\mathrm{q}} = \sigma_x + ir\sigma_z, \tag{2}$$

- $\mathcal{PT}$ symmetric operators have real eigenvalues with non-orthogonal eigenvectors and can be termed as observables. This regime entails a balance between gain and loss of the system at a characteristic recurrence time.
- Exceptional points - as the name suggests are exceptional. For a two-level system, eigenvectors at the exceptional points merge into one – a situation which is non-trivial.
- Non-$\mathcal{PT}$ symmetric operators have complex eigenvalues and is associated with loss in the system at a characteristic decay time.

$\mathcal{PT}$ symmetry violation on the Bloch sphere

$$H_{\mathrm{q}} = \sigma_x + ir\sigma_z$$



- Trajectories of the eigenvectors $|\psi_\pm\rangle$ for $r \in [0, 2]$.
- For $0 < r < 1$, the eigenvectors approach each other on the Bloch sphere.
- At $r = 0$, $H_q = \sigma_x$, with eigenvectors at $(\pm 1, 0, 0)$ shown with red and blue markers.
- At $r = 1$, the eigenvectors coalesce – green marker.
- For $r > 1$, eigenvectors asympotically approach $(0, 0, \pm 1)$.

Single-qubit non-Hermitian evolution

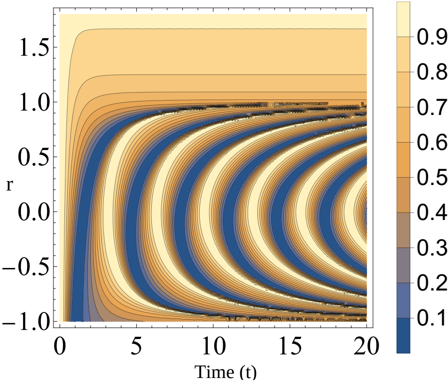
$$|0\rangle \longrightarrow \alpha_0(t)|0\rangle + \beta_0(t)|1\rangle,$$
$$|1\rangle \longrightarrow \alpha_1(t)|0\rangle + \beta_1(t)|1\rangle,$$
with $\gamma = \sqrt{1-r^2}$.

Normalized populations

$$p_0(t) = \frac{|\alpha_0(t)|^2}{|\alpha_0(t)|^2 + |\beta_0(t)|^2}$$
$$p_1(t) = \frac{|\beta_0(t)|^2}{|\alpha_0(t)|^2 + |\beta_0(t)|^2},$$

where,

$$\alpha_0(t) = \cos(\gamma t) + \frac{r}{\gamma}\sin(\gamma t)$$
$$\beta_0(t) = \alpha_1(t) = -\frac{i}{\gamma}\sin(\gamma t)$$
$$\beta_1(t) = \cos(\gamma t) - \frac{r}{\gamma}\sin(\gamma t).$$



For $r = 0$:

$$p_0(t) = \cos^2(t)$$
$$p_1(t) = \sin^2(t),$$

Physical realization of non-Hermitian evolution

- Aim: $i\frac{d}{dt}|\psi(t)\rangle_{\mathrm{q}} = H_{\mathrm{q}}|\psi(t)\rangle_{\mathrm{q}}$.
- Hermitian operator $\mathcal{H}_{\mathrm{a},q}(t)$ acting on the total qubit-ancilla Hilbert space using a Naimark dilation[3]

$$\mathcal{H}_{\mathrm{a,q}}(t) = \mathbb{I} \otimes \Lambda(t) + \sigma_y \otimes \Gamma(t), \tag{3}$$

- The dynamics under $\mathcal{H}_{\mathrm{a},q}(t)$ is determined by the Schrödinger equation

$$i\frac{d}{dt}|\Psi(t)\rangle_{\mathrm{a},q} = \mathcal{H}_{\mathrm{a},q}(t)|\Psi(t)\rangle_{\mathrm{a},q}, \tag{4}$$

whose solution is given by

$$|\Psi(t)\rangle_{\mathrm{a},q} = |0\rangle_{\mathrm{a}}|\psi(t)\rangle_{\mathrm{q}} + |1\rangle_{\mathrm{a}}|\tilde{\psi}(t)\rangle_{\mathrm{q}}, \tag{5}$$

where $|\psi(t)\rangle_{\mathrm{q}}$ is the solution.

[3] *Science*, **364**, 878-880 (2019).

Naimark dilated Hamiltonian

$$\mathcal{H}_{\mathrm{a,q}}(t) = \mathbb{I} \otimes \Lambda(t) + \sigma_y \otimes \Gamma(t), \tag{6}$$

with

$$\Lambda(t) = \left[H_{\mathrm{q}}(t) + i \frac{d\eta(t)}{dt} \eta(t) + \eta(t) H_{\mathrm{q}}(t) \eta(t) \right] \mathrm{M}^{-1}(t), \tag{7}$$

$$\Gamma(t) = i \left[H_{\mathrm{q}}(t) \eta(t) - \eta(t) H_{\mathrm{q}}(t) - i \frac{d\eta(t)}{dt} \right] \mathrm{M}^{-1}(t), \tag{8}$$

$$\eta(t) = (\mathrm{M}(t) - \mathbb{I})^{\frac{1}{2}}, \quad \text{and} \tag{9}$$

$$\mathrm{M}(t) = \mathrm{T} \exp\left[-i \int_0^t d\tau H_q^{\dagger}(\tau) \right] \mathrm{M}(0) \widetilde{\mathrm{T}} \exp\left[i \int_0^t d\tau H_q(\tau) \right], \tag{10}$$

$\eta(t)$ and $\mathrm{M}(t)$ are Hermitian operators; T and $\widetilde{\mathrm{T}}$ are time-ordering and anti-time-ordering operators respectively, and $\mathbb{I}$ is the 2×2 identity operator.

Initial conditions

$\mathrm{M}(t) - \mathbb{I}$ needs to be positive for all t. Thus, at $t = 0$, $\mathrm{M}(t)$ is chosen to be,

$$\mathrm{M}(t=0) = \mathrm{M}_0 = \frac{m_0}{\mu_{\min}} f \times \mathbb{I}, \tag{11}$$

$\mu_{\min}(t)$ and m_0 are minimum and maximum eigenvalues of $\mathrm{M}(t)$ in a given time interval $\{0, t\}$ and for arbitrary r. For any arbitrary r and t, $m_0/\mu_{\min} \geq 1$.
From Eq. 9 we have

$$\eta(t=0) = \left(\frac{m_0}{\mu_{\min}} f - 1\right)^{\frac{1}{2}} \times \mathbb{I} = \eta_0 \times \mathbb{I}. \tag{12}$$

Example: with $m_0 = 2$, $f = 1.01$; and time range $t \in [0, 8]$

- For $r = 0.6$: $\eta_0 = 1.7436$ and $\theta = 2.1001$ (radians).
- For $r = 1$: $\eta_0 = 16.1112$ and $\theta = 3.0176$ radians.
- For $r = 1.3$: $\mu_{\min}$ is obtained separately for various time intervals leading to different values of η_0 and θ.

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whose solution is given by

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where $|\psi(t)\rangle_{\mathrm{q}}$ is the solution.

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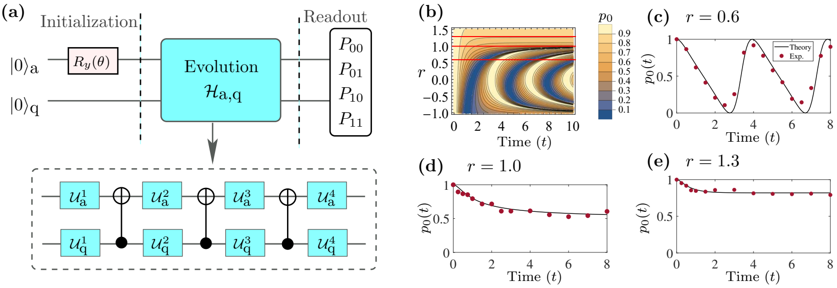
Single-qubit non-Hermitian evolution

Experimental demonstration of spontaneous $\mathcal{PT}$-symmetry breaking in a qubit, assisted by an ancilla



$$p_0(t) = \frac{P_{00}(t)}{P_{00}(t)+P_{01}(t)} \text{ and } p_1(t) = \frac{P_{01}(t)}{P_{00}(t)+P_{01}(t)}$$

Image source: S. Dogra, A. A. Melnikov, and G. S. Paraoanu, *Commun. Phys.* **4**, 26 (2021).

Distance between two quantum states

- Given: Two qrbitrary single-qubit states.
- Allow each of these to evolve under same unitary operation.
- Alternatively, locate both of these states on the Bloch sphere – as two points.

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- Can you think of a rotation of the Bloch sphere that can change the relative distance between these two points????

Distance between two quantum states

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- Allow each of these to evolve under same unitary operation.
- Alternatively, locate both of these states on the Bloch sphere – as two points.
- Can you think of a rotation of the Bloch sphere that can change the relative distance between these two points????
- **No!!!**

How about a general operation– not necessarily a rotation that can do the job...

Quantum state distinguishability

- Consider two single-qubit initial states $\rho_{1q}(0)$ and $\rho_{2q}(0)$.
- Each of these evolve under same Hamiltonian ($\mathcal{H}_q$) for same time t, such that the final states are: $\rho_{1q}(t)$ and $\rho_{2q}(t)$.
- Trace distance at an arbitrary time t:

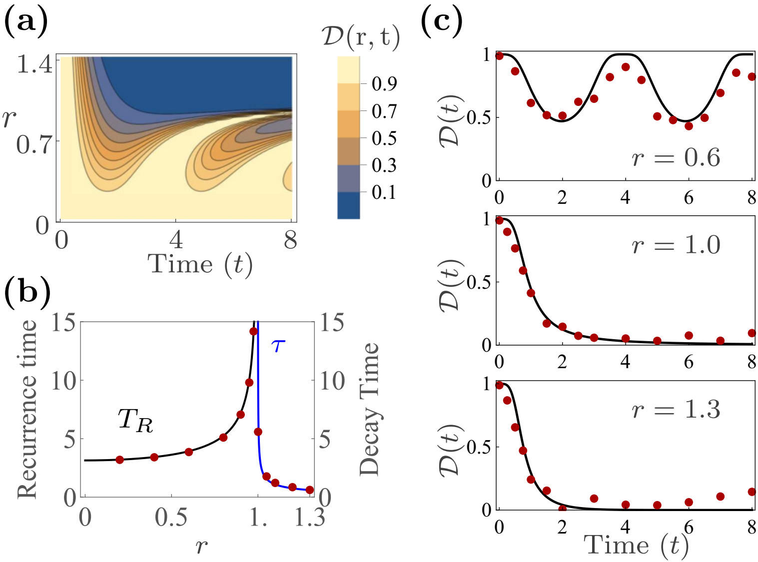
$$\mathcal{D}(\rho_{1q}(\mathrm{t}), \rho_{2q}(\mathrm{t})) = \frac{1}{2}\mathrm{tr}\sqrt{\rho_{\mathrm{diff}}(\mathrm{t})^{\dagger}\,\rho_{\mathrm{diff}}(\mathrm{t})},$$

  where $\rho_{\mathrm{diff}}(\mathrm{t}) = \rho_{1q}(\mathrm{t}) - \rho_{2q}(\mathrm{t})$ and $\rho_{iq}(t) = |\psi_i(t)\rangle_q\langle\psi_i(t)|_q$.
- In standard Hermitian quantum mechanics, $\mathcal{H}_q = \mathcal{H}_q^{\dagger}$ $\mathcal{D}(\rho_{1q}(\mathrm{t}), \rho_{2q}(\mathrm{t})) = \mathcal{D}(\rho_{1q}(0), \rho_{2q}(0))$

Designing a general protocol to distinguish arbitrary quantum states is not possible in Hermitian quantum mechanics.

Evolution of an arbitrary pair of states under a non-Hermitian operator can alter the distance between them, and may even make the arbitrary pair of quantum states orthogonal.

Quantum state distinguishability



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Recurrence and Decay times

The time-evolved state of the system, $\rho(t) = \frac{e^{-iH_q t}\rho(0)e^{iH_q^\dagger t}}{\mathrm{Tr}[e^{-iH_q t}\rho(0)e^{iH_q^\dagger t}]}$.

In the eigenbases of the Hamiltonian $\{|\psi_m\rangle, |\psi_n\rangle\}$,

$$\rho(t) = \frac{\sum_{mn} \rho_{mn} e^{-i(E_m - E_n)t} |\psi_m\rangle\langle\psi_n|}{\sum_{mn} \rho_{mn} e^{-i(E_m - E_n)t} \langle\psi_n|\psi_m\rangle}. \tag{16}$$

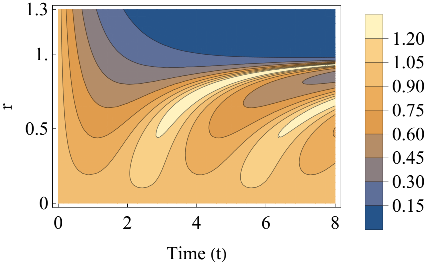
with $E_m - E_n = 2\sqrt{|1 - r^2|}$ and $r \neq 0$, norm: $N(t)$.

Recurrence time, T_R

$N(t + T_R) = N(t)$; $T_R = \frac{\pi}{\sqrt{1-r^2}}$

Decay time, τ_D

$\frac{1}{N(t+\tau_D)} = \frac{1}{eN(t)}$; $\tau_D = \frac{1}{2\sqrt{r^2-1}}$



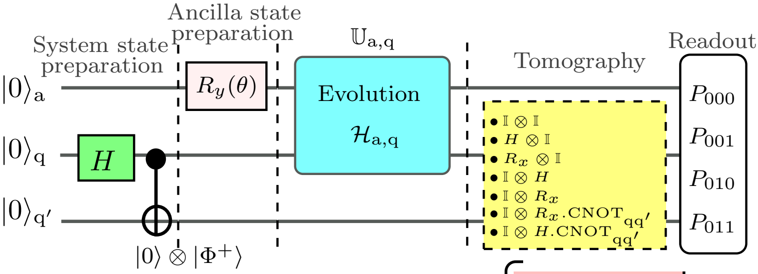
Entanglement monotonicity

Standard Hermitian quantum mechanics: entanglement between two parties cannot be increased under local operations.

Here we experimentally demonstrate an apparent violation of entanglement monotonicity in a two-qubit system, where one of the qubits evolve under a non-Hermitian Hamiltonian.

- Density operator of the system qubits in the post-selected subspace is $\rho^{(0)}_{\mathrm{q,q'}}$.
- Entanglement dynamics using concurrence as a measure given by $\mathcal{C}^{(0)}_{\mathrm{q,q'}} = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where λ_i's are the eigenvalues of the operator $\rho^{(0)}_{\mathrm{q,q'}}(\sigma_y \otimes \sigma_y)(\rho^{(0)}_{\mathrm{q,q'}})^\star(\sigma_y \otimes \sigma_y)$ written in decreasing order.

Experimental realization

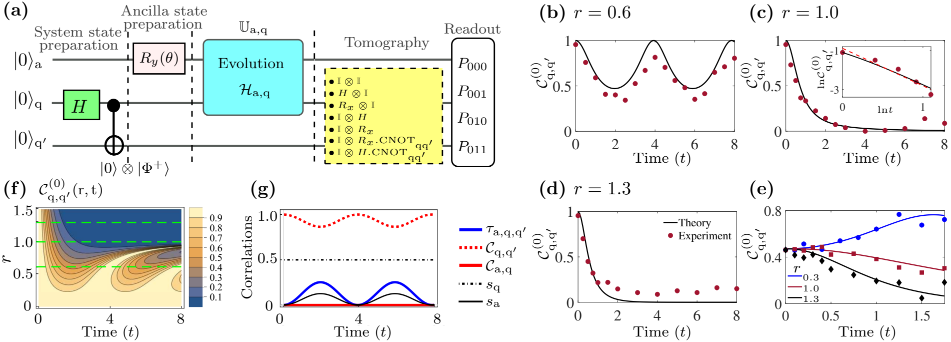


Postselection: $p^{(0)}_{\mathrm{j,k}} = \dfrac{p_{0,\mathrm{j,k}}}{\sum_{\mathrm{j,k}=0}^{1} p_{0,\mathrm{j,k}}}$.

1. $\rho^{exp}_{1,1}, \rho^{exp}_{2,2}, \rho^{exp}_{3,3}, \rho^{exp}_{4,4}$
2. $\mathrm{Re}(\rho^{\mathrm{exp}}_{1,3}), \mathrm{Re}(\rho^{\mathrm{exp}}_{2,4})$
3. $\mathrm{Im}(\rho^{\mathrm{exp}}_{1,3}), \mathrm{Im}(\rho^{\mathrm{exp}}_{2,4})$
4. $\mathrm{Re}(\rho^{\mathrm{exp}}_{1,2}), \mathrm{Re}(\rho^{\mathrm{exp}}_{3,4})$
5. $\mathrm{Im}(\rho^{\mathrm{exp}}_{1,2}), \mathrm{Im}(\rho^{\mathrm{exp}}_{3,4})$
6. $\mathrm{Re}(\rho^{\mathrm{exp}}_{1,4}), \mathrm{Re}(\rho^{\mathrm{exp}}_{2,3})$
7. $\mathrm{Im}(\rho^{\mathrm{exp}}_{1,4}), \mathrm{Im}(\rho^{\mathrm{exp}}_{2,3})$

$$\rho_{\mathrm{a,q,q'}} = \left[\begin{array}{c|c} |0\rangle_{\mathrm{a}}|\mathrm{j,k}\rangle_{\mathrm{q,q'}}\langle 0|_{\mathrm{a}}\langle \mathrm{m,n}|_{\mathrm{q,q'}} & |0\rangle_{\mathrm{a}}|\mathrm{j,k}\rangle_{\mathrm{q,q'}}\langle 1|_{\mathrm{a}}\langle \mathrm{m,n}|_{\mathrm{q,q'}} \\ \rho^{(\mathbf{0})}_{\mathrm{q,q'}} & \\ \hline |1\rangle_{\mathrm{a}}|\mathrm{j,k}\rangle_{\mathrm{q,q'}}\langle 0|_{\mathrm{a}}\langle \mathrm{m,n}|_{\mathrm{q,q'}} & |1\rangle_{\mathrm{a}}|\mathrm{j,k}\rangle_{\mathrm{q,q'}}\langle 1|_{\mathrm{a}}\langle \mathrm{m,n}|_{\mathrm{q,q'}} \\ & \rho^{(\mathbf{1})}_{\mathrm{q,q'}} \end{array}\right]$$

Entanglement monotonicity



Increase in concurrence as qubit q undergoes a local non-Hermitian evolution confirms the violation of entanglement monotonicity in the given post-selected subspace.

Image source: S. Dogra, A. A. Melnikov, and G. S. Paraoanu, *Commun. Phys.* **4**, 26 (2021).

Summary so far....

- Introduction to non-Hermitian quantum mechanics.
- Demonstration of $\mathcal{PT}$-symmetry breaking in a single qubit- Rabi oscillations is used as the signature.
- Arbitrary quantum states can be distinguished in the framework of non-Hermitian quantum mechanics.
- Demonstration of entanglement monotonicity in the post-selected subspace of a two-qubit system.
- Experiments are performed on IBM quantum experience.

Non-Hermitian non-PT-symmetric Hamiltonian

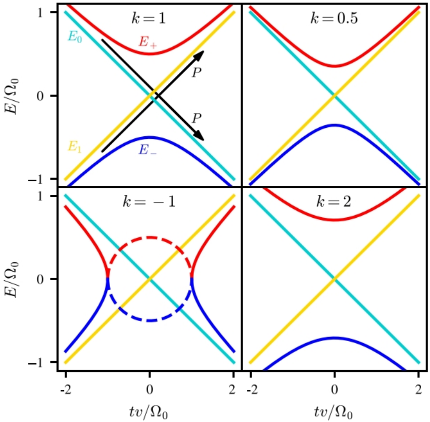
$$\hat{H}(t) = \frac{1}{2}\begin{bmatrix} -\varepsilon(t) & \Omega_0 \\ k\Omega_0 & \varepsilon(t) \end{bmatrix}$$
$$= \frac{1}{2}\left[\Omega_0\left(\frac{k+1}{2}\hat{\sigma}_x - i\frac{k-1}{2}\hat{\sigma}_y\right) - \varepsilon(t)\hat{\sigma}_z\right],$$

- For $k \neq 1$, $\hat{H}(t)$ is a non-Hermitian non-PT-symmetric time-dependent Hamiltonian.
- Rotating this Hamiltonian by $\pi/2$ around x-axis, we arrive at a PT-symmetric non-Hermitian Hamiltonian.

$$\hat{H}_{\mathrm{rot}} = \frac{1}{2}\left[\frac{k+1}{2}\Omega_0\hat{\sigma}_x + \varepsilon\hat{\sigma}_y - i\frac{k-1}{2}\Omega_0\hat{\sigma}_z\right]$$
$$= \frac{1}{2}\begin{bmatrix} -i\frac{k-1}{2}\Omega_0 & \frac{k+1}{2}\Omega_0 - i\varepsilon \\ \frac{k+1}{2}\Omega_0 + i\varepsilon & i\frac{k-1}{2}\Omega_0 \end{bmatrix},$$

Pseudo-Hermitian LZSM effect

$$\hat{H}(t) = \frac{1}{2}\begin{bmatrix} -\varepsilon(t) & \Omega_0 \\ k\Omega_0 & \varepsilon(t) \end{bmatrix}$$
$$= \frac{1}{2}\left[\Omega_0\left(\frac{k+1}{2}\hat{\sigma}_x - i\frac{k-1}{2}\hat{\sigma}_y\right) - \varepsilon(t)\hat{\sigma}_z\right],$$

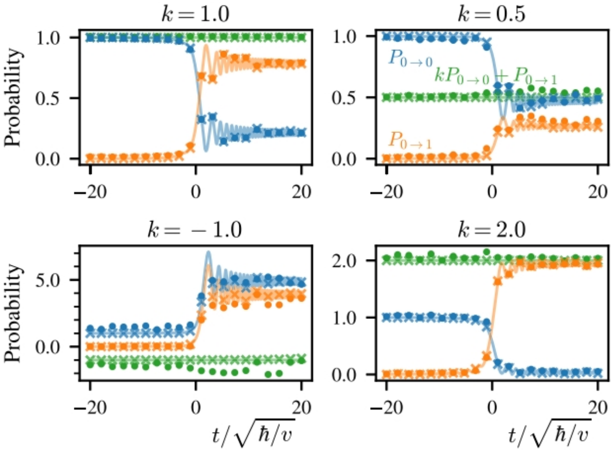


- Pseudo-Hermitian for real k, i.e.,

$$\hat{H}^\dagger = \hat{O}\hat{H}\hat{O}^{-1},$$

  where $\hat{O}$ is invertible Hermitian operator.
- Diabatic case ($\Omega_0 = 0$), eigenvalues $\pm\epsilon/2$
- Adiabatic case, $E_\pm = \pm\Delta E/2$, $\Delta E = \sqrt{k\Omega_0^2 + \epsilon^2}$

Pseudo-Hermitian LZSM effect



$$kP_{0\to 0}(t) + P_{0\to 1}(t) = k,$$

$$kP_{1\to 0}(t) + P_{1\to 1}(t) = 1,$$

KVANTTI group

- Feliks Kivela
- Artem Melnikov
- Sorin Paraoanu

Aalto University

- Quantum simulation of parity–time symmetry breaking with a superconducting quantum processor, S. Dogra, A. A. Melnikov, and G. S. Paraoanu, *C*ommun. Phys. **4**, 26 (2021).
- Quantum simulation of the pseudo-Hermitian Landau-Zener-Stückelberg-Majorana effect, F. Kivelä , S. Dogra , and G. S. Paraoanu, *P*hys. Rev. Res. **6**, 023246 (2024).

Thank you!